Krylov-based model reduction of second-order systems with proportional damping

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***Abstract*— In this note, we examine Krylov-based model reduction of second order systems where proportional damping is used to model energy dissipation. We give a detailed analysis of the distribution of system poles, and then, through a connection with potential theory, we are able to exploit the structure of these poles to obtain an optimal single shift strategy used in rational Krylov model reduction. We show that unlike the general case that requires usage of a second-order Krylov subspace structure, one can build up approximating subspaces satisfying all required conditions much more cheaply as direct sums of standard rational Krylov subspaces within the smaller component subspaces. Numerical examples are provided to illustrate and support the analysis.**

I. Introduction

In this paper, we examine the model reduction problem for second-order dynamical systems of the form

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{G}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) &= \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{1}$$

where $\mathbf{M}, \mathbf{G}, \mathbf{K} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{C} \in \mathbb{R}^{p\times n}$. In (1), $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ is the input force and $\mathbf{y}(t) \in \mathbb{R}^p$ is the output (measurements). Second order systems of the form (1) arise naturally in the analysis and modeling of structural vibration, electrical circuits, and micro-electro-mechanical systems; see, for example, [10], [22], [27], [9], [10], [20], [4], and references therein.

In many cases, the original system dimension n is too large for efficient simulation and control purposes. Therefore, the goal is to generate, for some $r \ll n$, an rth order reduced second-order system of the form

$$\begin{aligned} \mathbf{M}_r\ddot{\mathbf{x}}_r(t) + \mathbf{G}_r\dot{\mathbf{x}}_r(t) + \mathbf{K}_r\mathbf{x}_r(t) &= \mathbf{B}_r\mathbf{u}(t) \\ \mathbf{y}_r(t) &= \mathbf{C}_r\mathbf{x}_r(t) \end{aligned} \tag{2}$$

where $\mathbf{M}_r, \mathbf{G}_r, \mathbf{K}_r \in \mathbb{R}^{r\times r}$, $\mathbf{B}_r \in \mathbb{R}^{r\times m}$ and $\mathbf{C}_r \in \mathbb{R}^{p\times r}$ so that $\mathbf{y}_r(t)$ approximates $\mathbf{y}(t)$ for a wide range of inputs $\mathbf{u}(t)$.

We concentrate on a special case, namely second-order systems with proportional damping: the damping matrix $\mathbf{G}$ is given by $\mathbf{G} = \alpha\mathbf{M} + \beta\mathbf{K}$ for some choice of positive α and β with $\alpha\beta < 1$. We give a detailed analysis of the pole locations for such a system. We then review Krylov-based reduction and show that the specific structure of the problem allows one to use regular first-order Krylov subspaces in the reduction step. Based on the pole distribution, we propose an optimal single shift method for Krylov reduction. Numerical examples are given to illustrate and support the analysis.

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II. Modal Damping

Consider the forced vibration of an n degree–of–freedom mechanical structure modeled as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{G}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{b}u(t) \tag{3}$$

where $\mathbf{M}$ and $\mathbf{K}$ are positive definite symmetric matrices describing, respectively, mass and stiffness distributions throughout the structure. The matrix $\mathbf{G}$ describes energy loss due to internal damping or fluid viscosity and $u(t)$ is a time-dependent force applied along degrees-of-freedom specified in $\mathbf{b}$. Typically $\mathbf{M}$ and $\mathbf{K}$ are fairly well specified from an analytical model of the structure but $\mathbf{G}$ is poorly determined by such means. Qualitative modeling of energy dissipation in the structure leads naturally to the assumption that $\mathbf{G}$ is positive definite and symmetric as well. "Modal damping" assumes that the damped and undamped spatial modes are the same and amount to the assumption that $\mathbf{G}$ is simultaneously diagonalized by the same congruence transformation that simultaneously diagonalizes $\mathbf{M}$ and $\mathbf{K}$. Indeed, $\mathbf{K}$ and $\mathbf{M}$ may be simultaneously diagonalized with a nonsingular congruence transformation:

$$\mathbf{X}^T\mathbf{K}\mathbf{X} = \text{ diag}(\omega_\ell^2) \quad \text{ and } \quad \mathbf{X}^T\mathbf{M}\mathbf{X} = \mathbf{I}$$

for some nonsingular $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n]$ where $\mathbf{x}_\ell$ is an eigenvector $\mathbf{K}\mathbf{x}_\ell = \omega_\ell^2\mathbf{M}\mathbf{x}_\ell$.

We assume then that the damping matrix $\mathbf{G}$ is also diagonalized by $\mathbf{X}$: $\mathbf{X}^T\mathbf{G}\mathbf{X} = \text{diag}(2\xi_\ell\omega_\ell)$, where a different damping ratio ξ_ℓ could (in principle) be specified for each $i = 1, \ldots, n$. This flexibility in specifying damping ratios is captured for us through a damping function, $g(z)$, that is real analytic on the positive halfline and such that for real $z > 0$, $g(z) > 0$. One can arrange in particular that $g(\omega_\ell^2) = 2\xi_\ell\omega_\ell$ or more generally that

$$\mathbf{G} = \mathbf{M}\mathbf{X}\ \text{diag}(g(\omega_\ell^2))\mathbf{X}^T\mathbf{M} = \mathbf{M}\ g(\mathbf{M}^{-1}\mathbf{K})$$

Common choices for $g(z)$ include $g(z) = 2\xi\sqrt{z}$ (fractional damping) or $g(z) = \alpha + \beta z$ (proportional damping).

$\mathbf{X}$ diagonalizes the quadratic pencil associated with (3):

$$\mathbf{X}^T\left(\lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K}\right)\mathbf{X} = \text{diag}(\lambda^2 + \lambda g(\omega_\ell^2) + \omega_\ell^2)$$

Letting ω represent any of the undamped frequencies $\omega_1, \omega_2, \ldots$, the associated damped eigenvalue λ satisfies

$$\lambda^2 + \lambda\ g(\omega^2) + \omega^2 = 0.$$

for some $\omega = \omega_\ell$. That is,

$$\lambda = -\frac{g(\omega^2)}{2} \pm \sqrt{\left(\frac{g(\omega^2)}{2}\right)^2 - \omega^2}.$$

Damped eigenvalues will have nontrivial imaginary parts (producing oscillatory modes) when the discriminant is negative, which happens precisely for ω in the range determined by:$\{\omega \mid g(\omega^2) < 2\omega\}$. For such ω, λ evidently can be written as

$$\lambda = -\frac{g(\omega^2)}{2} \pm \imath\sqrt{\omega^2 - \left(\frac{g(\omega^2)}{2}\right)^2}$$

and immediately one has that for any real ρ:

$$\begin{aligned} |\lambda + \rho|^2 &= \left(\rho - \frac{g(\omega^2)}{2}\right)^2 + \left(\omega^2 - \left(\frac{g(\omega^2)}{2}\right)^2\right) \\ &= \rho^2 - \rho\, g(\omega^2) + \omega^2. \end{aligned} \tag{4}$$

In particular, $|\lambda| = \omega$, independent of the choice of damping function, $g(z)$.

Proposition 1: In the case of proportional damping, all damped eigenvalues λ with nontrivial imaginary parts lie on a circle centered at $-\frac{1}{\beta}$ with radius $\frac{\sqrt{1-\alpha\beta}}{\beta}$.

PROOF: Eigenvalues will have nontrivial imaginary parts when ω is in the range: $\{\omega \mid \alpha + \beta\omega^2 < 2\omega\}$, i. e.,

$$\frac{1}{\beta}\left(1 - \sqrt{1-\alpha\beta}\right) < \omega < \frac{1}{\beta}\left(1 + \sqrt{1-\alpha\beta}\right). \tag{5}$$

Setting $\rho = \frac{1}{\beta}$ in (4) we see that $|\lambda + \frac{1}{\beta}|^2 = \frac{1-\alpha\beta}{\beta^2}$. $\square$

Notice that the circle containing eigenvalues associated with oscillatory modes depends only on the damping parameters α and β and is independent of $\mathbf{M}$ and $\mathbf{K}$. However the distribution of damped eigenvalues around this circle will depend on $\mathbf{M}$ and $\mathbf{K}$, and, in particular, on the distribution of the undamped frequencies, ω_ℓ. Indeed, we can write the damped eigenvalues associated with oscillatory modes as

$$\lambda = -\frac{1}{\beta} + \frac{\sqrt{1-\alpha\beta}}{\beta}\, e^{\imath\theta}, \tag{6}$$

where θ depends on an undamped frequency ω:

$$\cos(\theta(\omega)) = \frac{2 - \alpha\beta - (\beta\omega)^2}{2\sqrt{1-\alpha\beta}}$$

III. KRYLOV-BASED MODEL REDUCTION

Consider the following single-input/single-output linear time-invariant dynamical system $\mathbf{H}(s)$ in generalized state-space form:

$$\mathbf{H}(s) : \left\{ \begin{array}{rcl} \mathcal{E}\,\dot{\mathbf{q}}(t) &=& \mathcal{A}\mathbf{q}(t) + \mathcal{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) &=& \mathcal{C}\,\mathbf{q}(t) \end{array} \right. \tag{7}$$

where $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{N\times N}$ and $\mathcal{B}, \mathcal{C}^T \in \mathbb{R}^N$. The transfer function of (7) is given by

$$\mathbf{H}(s) = \mathcal{C}\,(s\mathcal{E} - \mathcal{A})^{-1}\,\mathcal{B}.$$

(Note that both the underlying dynamical system and its transfer function are denoted by the same $\mathbf{H}(s)$.) The goal of Krylov-based model reduction is to find a reduced-order dynamical system $\mathbf{H}_r(s)$ by projecting (7) in such a way that $\mathbf{H}_r(s)$ interpolates $\mathbf{H}(s)$ as well as a certain number of its derivatives ('moments') at selected points σ_k in the complex plane, i.e., choose matrices $\mathcal{V} \in \mathbb{R}^{N\times r}$ and $\mathcal{Z} \in \mathbb{R}^{n\times r}$ so that $\mathcal{V}^T\mathcal{Z} = \mathbf{I}_r$ and

$$\begin{aligned} \underbrace{\mathcal{Z}^T\mathcal{E}\mathcal{V}}_{\mathcal{E}_r}\,\dot{\mathbf{q}}_r(t) &= \underbrace{\mathcal{Z}^T\mathcal{A}\mathcal{V}}_{\mathcal{A}_r}\mathbf{q}(t) + \underbrace{\mathbf{Z}^T\mathcal{B}}_{\mathcal{B}_r}\,\mathbf{u}(t) \\ \mathbf{y_r}(t) &= \underbrace{\mathcal{C}\mathcal{V}}_{\mathcal{C}_r}\,\mathbf{q}(t) \end{aligned} \tag{8}$$

and

$$\left.\frac{d^j\mathbf{H}(s)}{ds^j}\right|_{s=\sigma_k} = \left.\frac{d^j\mathbf{H}_r(s)}{ds^j}\right|_{s=\sigma_k}$$

for $k = 1, \ldots, K$ and for $j = 0, \ldots, J-1$. Here K is the number of interpolation points σ_k and J is the number of moments to be matched at each σ_k. This problem is sometimes called multi-point rational interpolation by projection. In a projection framework, this problem was first treated by Skelton *et. al.* in [11], [29], [28]. Grimme [17] showed how one can obtain the required projection in a numerically efficient way using the rational Krylov method of Ruhe [23].

For a matrix $\mathcal{F} \in \mathbb{C}^{N\times N}$, a vector $\mathbf{g} \in \mathbb{C}^N$, and a point $\sigma \in \mathbb{C}$, define the Krylov space (of generic dimension J):

$$\mathcal{K}(\mathcal{F}, \mathbf{g}) = \text{span}\{\,\mathbf{g},\, \mathcal{F}\mathbf{g},\, \mathcal{F}^2\mathbf{g},\, \cdots,\, \mathcal{F}^{J-1}\mathbf{g}\,\}$$

The following theorem, presented in [17], connects multi-point rational interpolation with Krylov projections:

Theorem 1: If $\mathcal{V}$ and $\mathcal{Z}$ satisfy $\mathcal{V}^T\mathcal{Z} = \mathbf{I}_r$,

$$\begin{aligned} &\text{Ran}(\mathcal{V}) = \text{span}\left\{\mathcal{K}(\mathcal{F}_1, \mathbf{g}_1),\, \cdots,\, \mathcal{K}(\mathcal{F}_K, \mathbf{g}_K)\right\}, \text{ and} \\ &\text{Ran}(\mathcal{Z}) = \text{span}\left\{\mathcal{K}(\mathcal{F}_{K+1}, \mathbf{g}_{K+1}), \cdots, \mathcal{K}(\mathcal{F}_{2K}, \mathbf{g}_{2K})\right\}, \end{aligned}$$

for

$$\begin{aligned} &\mathcal{F}_i = (\sigma_i\mathcal{E} - \mathcal{A})^{-1}\mathcal{E}, \quad \mathbf{g}_i = (\sigma_i\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}, \\ &\qquad\qquad\qquad\qquad \text{for } i = 1, \ldots, K \\ &\mathcal{F}_i = (\sigma_i\mathcal{E} - \mathcal{A})^{-T}\mathcal{E}^T, \quad \mathbf{g}_i = (\sigma_i\mathcal{E} - \mathcal{A})^{-T}\mathcal{C}^T, \\ &\qquad\qquad\qquad\qquad \text{for } i = K+1, \ldots, 2K \end{aligned}$$

then the reduced order model $\mathbf{H}_r(s)$ given in (8) has order JK and matches J moments of $\mathbf{H}(s)$ at each of the $2K$ interpolation points σ_k, $k = 1, \ldots, 2K$: $\mathbf{H}_r(s)$ interpolates $\mathbf{H}(s)$ and its first $J-1$ derivatives at each σ_k.

As a consequence of this result, Krylov-based model reduction requires selection of suitable interpolation points and then construction of matrices $\mathcal{V}$ and $\mathcal{Z}$ as above. Efficient implementation is discussed in [17]. For more details on Krylov based model reduction, see [14], [15], [17], [19], [1], [16], [2].

A. Second-order systems with proportional damping

Consider now Krylov-based model reduction of a single-input/single-output second-order system where proportional damping is used to model damping, i.e., we have a second order system of the form

$$\begin{aligned} &\mathbf{M}\ddot{\mathbf{x}}(t) + (\alpha\mathbf{M} + \beta\mathbf{K})\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t), \\ &\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \end{aligned} \tag{9}$$

where $\mathbf{M}, \mathbf{K} \in \mathbb{R}^{n\times n}$, $\mathbf{B}$, $\mathbf{C}^T \in \mathbb{R}^n$ and α, β are proportional damping coefficients with α, $\beta > 0$ and $\alpha\beta < 1$. Let $\mathbf{q}(t) = [\ x^T(t) \quad \dot{x}^T(t)\ ]^T$. Then, a generalized state-space realization of the corresponding first-order linearized model is given by

$$\mathcal{E}\dot{\mathbf{q}} = \mathcal{A}\mathbf{q}(t) + \mathcal{B}u(t), \quad \mathbf{y}(t) = \mathcal{C}\mathbf{q}(t) \tag{10}$$

where

$$\mathcal{E} = \begin{bmatrix} I & 0 \\ 0 & \mathbf{M} \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & I \\ -\mathbf{K} & -\alpha\mathbf{M} - \beta\mathbf{K} \end{bmatrix} \tag{11}$$

$$\mathcal{B} = \begin{bmatrix} 0 \\ \mathbf{B} \end{bmatrix}, \text{ and } \mathcal{C} = [\ \mathbf{C} \quad 0\ ]. \tag{12}$$

The transfer function of this system is given by

$$\mathbf{H}(s) = \mathbf{C}(s^2\mathbf{M} + s\mathbf{G} + \mathbf{K})^{-1}\mathbf{B} = \mathcal{C}(s\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}. \tag{13}$$

To obtain a reduced-order model that matches the moments of the original model (10) at σ, one can use Theorem 1 to specify $\mathcal{V}$ and $\mathcal{Z}$ and obtain a reduced model as in (8). However, this reduction is within the first-order system framework and might destroy the original second-order system structure: it wil not always be possible to obtain a reduced-order system corresponding to a second-order system of the form (9). Even when this is possible, one typically cannot guarantee that properties such as positive definite reduced-order mass and stiffness matrices, will be preserved. Alternatively, one may apply reduction directly in the second-order system framework, i.e. find a matrix $\mathbf{W} \in \mathbb{R}^{n\times r}$ such that $\mathbf{W}^T\mathbf{W} = \mathbf{I}_r$ and such that the associated reduced-order model given by

$$\mathbf{M}_r\ddot{\mathbf{x}}_r(t) + \mathbf{G}_r\dot{\mathbf{x}}_r(t) + \mathbf{K}_r\mathbf{x}_r(t) = \mathbf{B}_r\mathbf{u}(t) \tag{14}$$

$$\mathbf{y}_r(t) = \mathbf{C}_r\mathbf{x}_r(t) \tag{15}$$

with

$$\mathbf{M}_r = \mathbf{W}^T\mathbf{M}\mathbf{W}, \ \mathbf{G}_r = \mathbf{W}^T\mathbf{G}\mathbf{W}, \tag{16}$$

$$\mathbf{K}_r = \mathbf{W}^T\mathbf{K}\mathbf{W}, \mathbf{B}_r = \mathbf{W}^T\mathbf{B}, \text{ and } \mathbf{C}_r = \mathbf{C}\mathbf{W} \tag{17}$$

satisfies the desired interpolation conditions. See [4], [6], [26], [3], [8], [9], [13] for some recent work towards these goals. It was shown in [4], [3] that Krylov-based model reduction can be done directly in the second-order system framework as in (14-17) above by introducing the so-called *second-order Krylov subspaces* and *second-order Arnoldi procedure*. These methods use in effect, a two-stage recurrence in $\mathbb{R}^n$ to generate the effect of the usual one-stage Krylov recurrence in $\mathbb{R}^{2n}$. In the sequel, we will observe that in special case of proportional damping, Krylov-based model reduction can be done directly in the second-order system framework using usual only a one-stage Krylov recurrence (in $\mathbb{R}^{2n}$).

Even though our final model reduction will be done in a second-order system framework (9), moment matching is best discussed in a first-order system framework (10). In order to match the first r moments of (10) at σ (which are same as those of (9)), then due to Theorem 1 one needs to construct the Krylov subspace:

$$\mathcal{V} = \text{span}\left\{\mathbf{g}, \mathcal{F}\mathbf{g}, \cdots, \mathcal{F}^{r-1}\mathbf{g}\right\} \tag{18}$$

where

$$\mathcal{F} = (\sigma\mathcal{E} - \mathcal{A})^{-1}\mathcal{E} \quad \text{and} \quad \mathbf{g} = (\sigma\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}. \tag{19}$$

and $\mathcal{E}, \mathcal{A}$ and $\mathcal{B}$ are as given in (11) and (12). Next, we state the main result of this section:

Theorem 2: For the linearized first-order system framework (10)–(12), the associated Krylov subspace $\mathcal{V}$ defined in (18), that induces interpolation at σ may be decomposed as

$$\mathcal{V} \subset \mathcal{W} \oplus \mathcal{W} \tag{20}$$

where $\mathcal{W} = \mathcal{K}(\mathbf{K}_\sigma^{-1}\mathbf{M}, \mathbf{K}_\sigma^{-1}\mathbf{B})$ with $\mathbf{K}_\sigma = \sigma^2\mathbf{M} + \sigma(\alpha\mathbf{M} + \beta\mathbf{K}) + \mathbf{K}$.

Hence, one can apply model reduction directly in the second-order system framework as in (16-17) using an orthonormal basis for $\mathcal{W}$: $\text{Ran}(\mathbf{W}) = \mathcal{W}$ and $\mathbf{W}^T\mathbf{W} = \mathbf{I}_r$. The resulting second-order system matches r moments of the full-order model (9) at the interpolation point σ. Moreover, if $\mathbf{M}$ and $\mathbf{K}$ are symmetric and $\mathbf{B} = \mathbf{C}^T$, then the reduced-order model matches $2r$ moments at σ.

The decomposition in (20) corresponds to the separation of degrees of freedom into displacement components $\mathbf{x}$ and velocity components $\dot{\mathbf{x}}$.

PROOF: We only sketch the proof here: The first vector required to specify the Krylov subspace $\mathcal{V}$ is

$$\mathbf{g} = (\sigma\mathcal{E} - \mathcal{A})^{-1}\mathcal{B} = \begin{bmatrix} \mathbf{K}_\sigma^{-1}\mathbf{B} \\ \sigma\mathbf{K}_\sigma^{-1}\mathbf{B} \end{bmatrix}. \tag{21}$$

and $\mathbf{K}_\sigma^{-1}\mathbf{B} \in \mathcal{W}$. Now we follow an induction argument. Let $\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}$ be such that $\mathbf{v}_1, \mathbf{v}_2$ are in a Krylov space of order $p \geq 1$ generated by $\mathbf{K}_\sigma^{-1}\mathbf{M}$ on the starting vector $\mathbf{K}_\sigma^{-1}\mathbf{B}$. Consider the outcome of $\bar{\mathbf{v}} = (\sigma\mathcal{E} - \mathcal{A})^{-1}\mathcal{E}\mathbf{v}$. Elementary manipulations yield

$$\bar{\mathbf{v}} = \begin{bmatrix} \bar{\mathbf{v}}_1 \\ \bar{\mathbf{v}}_2 \end{bmatrix} = \begin{bmatrix} \kappa_1\mathbf{v}_1 + \kappa_2\mathbf{K}_\sigma^{-1}\mathbf{M}\mathbf{v}_1 + \kappa_3\mathbf{K}_\sigma^{-1}\mathbf{M}\mathbf{v}_2 \\ \kappa_4\mathbf{v}_1 + \kappa_5\mathbf{K}_\sigma^{-1}\mathbf{M}\mathbf{v}_1 + \kappa_6\mathbf{K}_\sigma^{-1}\mathbf{M}\mathbf{v}_2 \end{bmatrix}$$

where κ_i, $i = 1, \ldots, 6$ are constants which depends on β, α and σ. Thus $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2$ are in a Krylov space of order $p+1$ generated by $\mathbf{K}_\sigma^{-1}\mathbf{M}$ on the starting vector $\mathbf{K}_\sigma^{-1}\mathbf{B}$.

The second part of the theorem, i.e., the moment matching part, follows from the facts that the second-order reduction in (16)-(17) using $\mathbf{W}$ of with $\text{Ran}(\mathbf{W}) = \mathcal{W}$ amounts to reducing the first-order matrices $\mathcal{A}$, $\mathcal{E}$, $\mathcal{B}$, and

$\mathcal{C}$ with $\mathcal{V} = \mathcal{Z} = \begin{bmatrix} \mathbf{W} & 0 \\ 0 & \mathbf{W} \end{bmatrix}$. If $\mathbf{M}$ and $\mathbf{K}$ are symmetric, and $\mathbf{B} = \mathbf{C}^T$, the number of moments matched doubles. $\square$

Remark 1: The main difference from the results of [3], [4] is that in the particular case of proportional damping, the decomposition of the Krylov subspace needed for moment matching as shown in (20) allows the problem to be resolved by generating a *regular first-order Krylov subspace* $\mathbf{W}$: the concept of second-order Krylov subspace is not required.

Remark 2: If $\sigma = 0$, the solution strategy requires computing a basis for

$$\mathcal{W} = \text{span}\{\mathbf{K}^{-1}\mathbf{B}, \cdots, (\mathbf{K}^{-1}\mathbf{M})^{r-1}\mathbf{K}^{-1}\mathbf{B}\}$$

On the other hand, if $\sigma = \infty$, then

$$\mathcal{W} = \text{span}\{\mathbf{M}^{-1}\mathbf{B}, \cdots, (\mathbf{M}^{-1}\mathbf{K})^{r-1}\mathbf{M}^{-1}\mathbf{B}\}$$

Remark 3: Results can be naturally extended to the case where multiple interpolation points σ_i, $i = 1, \cdots, k$ are chosen. In this case, $\mathbf{W}$ should span the union of the Krylov subspaces corresponding to each interpolation point σ_i.

IV. Shift selection

Based on the pole distribution of second-order systems with proportional damping, we propose a (optimal) shift selection strategy for Krylov-based model reduction of such systems.

As discussed in [5], ideal interpolation points (the rational Krylov shifts) are chosen to be reflections of eigenvalues across the imaginary axis. Any configuration of shifts that will produce minimal $\mathcal{H}_\infty$ error must force the difference of the true and reduced order transfer function to be constant or nearly so along the imaginary axis. This can be done by forcing interpolation points to be symmetrically distributed across the imaginary axis with respect to system poles. There is an electrostatic analogy in this analysis that associates both the system poles and the system interpolation points with point charges (but of opposite sign) and the rate of error reduction then is related to the potential difference generated by the charge configuration. A key feature of the optimal shift distrubution is that the imaginary axis must be an equipotential.

In some cases, indeed in many problems arising in technical mechanics, the distribution of eigenvalues and the ideally paired interpolation points will often come very close to an "equilibrium charge distribution" over the twin circles on which the eigenvalues are constrained to lie and the interpolation points are induced to lie. The configuration of interpolation points can be viewed as a distributed charge distribution and be effectively replaced with a single shift (that is, a single "equivalent charge") at which successively higher moments are then matched.

For $\gamma > 0$, consider

$$\zeta(z) = \frac{z+\gamma}{z-\gamma}$$

$\zeta(z)$ maps the left half plane to the interior of the unit disk; the imaginary axis onto the unit circle; and maps the point $z = -\gamma$ to 0. If $\gamma(2-\beta\gamma) > \alpha$ (so that $\left|\frac{1}{\beta} - \gamma\right| < \frac{\sqrt{1-\alpha\beta}}{\beta}$,) then the image, $\zeta(\lambda)$, of all complex eigenvalues will lie on a single circle within the unit disk:

$$\left|\zeta(\lambda) - \frac{\alpha - \beta\gamma^2}{\alpha + 2\gamma + \beta\gamma^2}\right| = \frac{2\sqrt{1-\alpha\beta}\,\gamma}{\alpha + 2\gamma + \beta\gamma^2} < 1.$$

If $\gamma = \sqrt{\frac{\alpha}{\beta}}$ then the circle is itself centered within the unit disk and has radius $\sqrt{\frac{1-\sqrt{\alpha\beta}}{1+\sqrt{\alpha\beta}}}$. Hence, for the case where the pole distribution of the proportionally damped second-order system corresponds to an exact condenser distribution centered at $-\frac{1}{\beta}$ and with radius $\frac{\sqrt{1-\alpha\beta}}{\beta}$,

$$\sigma_* = \sqrt{\frac{\alpha}{\beta}} \tag{22}$$

will be the *optimal* single interpolation for a Krylov based model reduction. Even when the distribution of poles is not an exact condenser distribution, they still lie on the same circle and this shift choice is expected to being close to optimal. Examples below show that this is indeed the case. Note that computation of σ_* requires only α and β, which are design parameters. Perhaps most significantly, the corresponding Krylov-based reduction will require only one linear solve.

A. A second-order system with exact condenser distribution

We can produce matrices $\mathbf{M}$ and $\mathbf{K}$ such that the second-order system with proportional damping (9) has *an exact condenser pole distribution* and therefore $\sigma_* = \sqrt{\frac{\alpha}{\beta}}$ will be a true optimal shift.

Pick $\alpha,\ \beta \in (0,1)$ and consider the matrix pair defined by

$$\mathbf{K} = \frac{\alpha}{\beta}\begin{bmatrix} \frac{2-\sqrt{1-\alpha\beta}}{\sqrt{1-\alpha\beta}} & -1 & 0 & \cdots & \\ -1 & \frac{2}{\sqrt{1-\alpha\beta}} & & & \\ 0 & & \ddots & & 0 \\ \vdots & & & \frac{2}{\sqrt{1-\alpha\beta}} & -1 \\ & & 0 & -1 & \frac{2-\sqrt{1-\alpha\beta}}{\sqrt{1-\alpha\beta}} \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} \frac{2+\sqrt{1-\alpha\beta}}{\sqrt{1-\alpha\beta}} & 1 & 0 & \cdots & \\ 1 & \frac{2}{\sqrt{1-\alpha\beta}} & & & \\ 0 & & \ddots & & 0 \\ \vdots & & & \frac{2}{\sqrt{1-\alpha\beta}} & 1 \\ & & 0 & 1 & \frac{2+\sqrt{1-\alpha\beta}}{\sqrt{1-\alpha\beta}} \end{bmatrix}.$$

The eigenvalues of the pencil $\mathbf{K}\mathbf{x} = \omega^2\mathbf{M}\mathbf{x}$ are

$$\omega_\ell^2 = \left(\frac{\alpha}{\beta}\right)\frac{1 - \sqrt{1-\alpha\beta}\cos\left(\frac{2\ell-1}{2n+1}\pi\right)}{1 + \sqrt{1-\alpha\beta}\cos\left(\frac{2\ell-1}{2n+1}\pi\right)}$$

It follows from this last expression that poles of a second-order dynamical system with mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ as above, with damping $\mathbf{G} = \alpha\mathbf{M} + \beta\mathbf{K}$ will

correspond to an exact condenser distribution, and hence σ_* as in (22) will be the corresponding optimal single shift for Krylov-based model reduction.

V. Examples

A. Exact Condenser Distribution

For this example, $\mathbf{M}$ and $\mathbf{K}$ matrices are chosen as in Section IV-A, and therefore, the resulting second-order system with proportional damping has an exact condenser pole distribution. Damping parameters α and β are chosen as $\alpha = \beta = 0.05$. Distribution of the system poles is depicted in Figure 1. Input and output matrices are $\mathbf{B} = \mathbf{C}^T = [1\ 0\ 0\ \cdots\ 0]^T$. Order of the system is $n = 2000$ and we reduce the order to $r = 30$ using a single shift. We vary the shift between 0 and 10^3, and plot $\mathcal{H}_\infty$ norm of the resulting error systems vs the interpolation points in Figure 2. As the figure clearly indicates, as expected $\sigma_* = \sqrt{\alpha/\beta} = 1$ is the optimal shift.

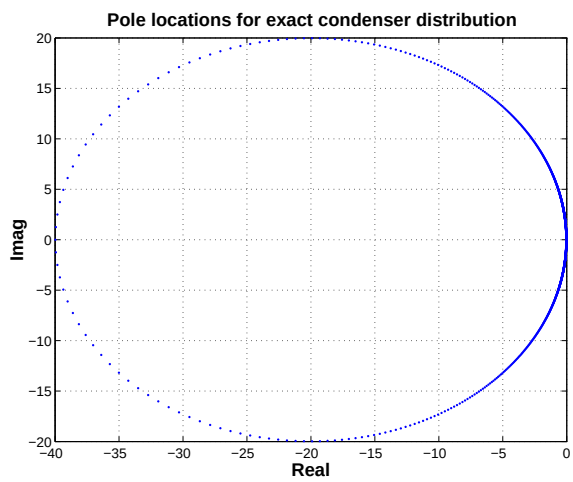

Fig. 1. Location of the poles for the exact condenser distribution

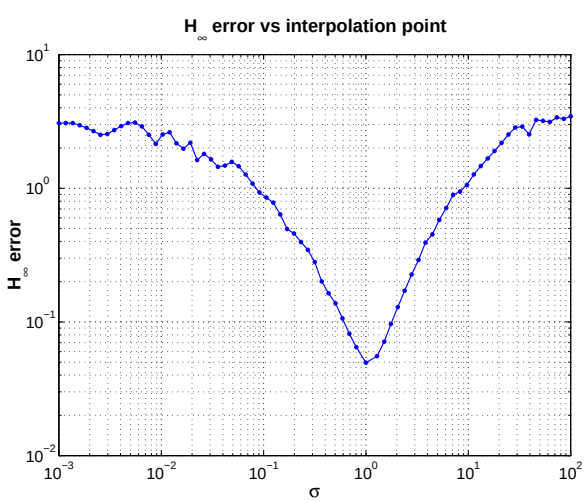

Fig. 2. $\mathcal{H}_\infty$ norm of the error system for the exact condenser distribution

B. A 1-D Beam Model

The full-order model represents the second-order dynamics of a 1-D beam with state-space dimensions $M, K \in \mathbb{R}^{200\times 200}$. Damping is modeled as a proportional damping with coefficients $\alpha = 1/10$ and $\beta = 1/500$. Input is a point force applied to the state $x(1)$ and the output is the displacements at $x(200)$. We reduce the order to $r = 5, 10, 15$ using Krylov projection with a single shift. From our analysis, we expect the optimal shift to be $\sigma_* = \sqrt{\frac{\alpha}{\beta}} = 7.0711$. Figure 3 below shows the $\mathcal{H}_\infty$ norm of the resulting error system as the interpolation point σ varies. A closer examination of the figure illustrates that the observed single optimal shift is around 7.1 which is very close to our estimation 7.07. The small difference is due to deviation from the condenser pole distribution as shown in Figure 4.

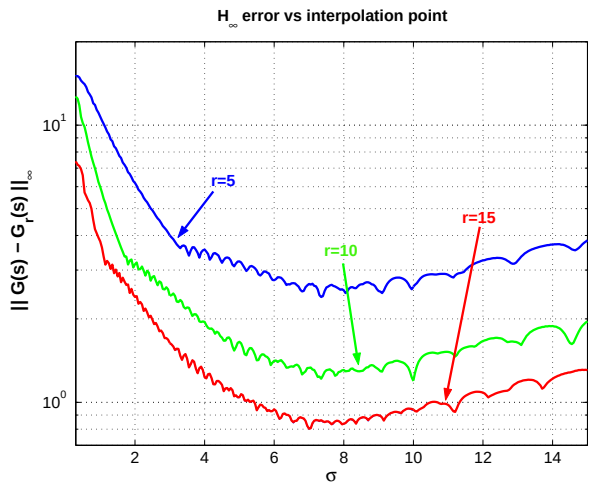

Fig. 3. $\mathcal{H}_\infty$ norm of the error system for the 1-D Beam Model

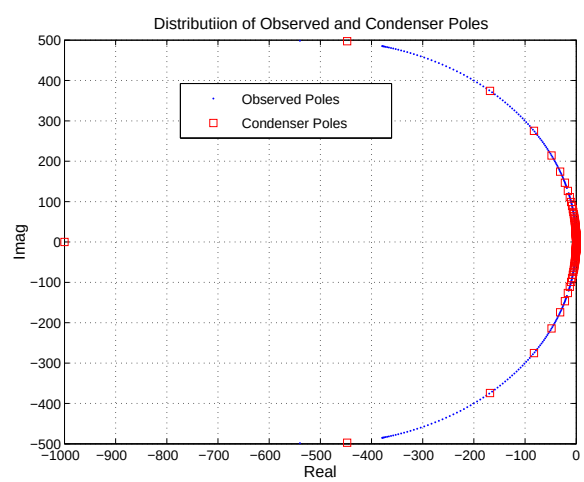

Fig. 4. Pole distribution for the 1-D Beam Model

VI. Conclusions

We have studied Krylov-based model reduction of second-order dynamical systems with proportional damping. We showed the specific damping structures allows one to apply Krylov-based model reduction directly in the second-order framework using *regular first-order* Krylov subspace iterations. Moreover, based on a detailed analysis of the pole locations for these systems, we are able to provide an optimal or near optimal single shift selection for the Krylov reduction. Numerical examples have illustrated the effectiveness of the proposed shift selection.

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