

Krylov-Based Controller Reduction for Large-Scale Systems

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Abstract—In this paper, we introduce a computationally efficient controller reduction approach using a rational Krylov method. We show that a reduced-order controller obtained via a Krylov projection is guaranteed to match the desired full-order closed loop system response at shifts used in the Krylov reduction of the controller. Two different shift selection strategies are proposed. A numerical example illustrates the effectiveness of the proposed approach.

I. INTRODUCTION

Consider an n^{th} order linear time-invariant continuous-time plant $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$ with transfer function $G(s) = C(sI - A)^{-1}B + D$, and an n_κ^{th} order stabilizing controller $K(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right]$ with transfer function $K(s) = C_K(sI - A_K)^{-1}B_K + D_K$. Many control design methodologies, such as LQG and \mathcal{H}_∞ methods, lead ultimately to controllers whose order is generically as high as the order of the plant, see [10], [11] and references therein. Thus large-scale systems lead to large-scale controllers. Typically, high-order controllers are undesirable in real-time applications for three main reasons [2], [1], [10]: (i) *Complex hardware*: A large-scale controller typically requires complex hardware and a large investment in implementation; (ii) *Degraded accuracy*: Due to ill-conditioning in large-scale computations, it might not be possible to operate such a controller within the required accuracy margins; and (iii) *Degraded computational speed*: The time needed to compute the output response for a complex controller might be too long, possibly longer than the system sampling time, yielding ineffective and potentially destructive feedback inputs. Therefore, we seek to obtain a reduced order controller $K_r(s)$ of order r with $r \ll n_\kappa$ to replace $K(s)$. For simplicity, we assume that $G(s)$ and $K(s)$ are each single-input/single-output (SISO) systems; i.e. $B, C^T \in \mathbb{R}^n$ and $B_K, C_K^T \in \mathbb{R}^{n_\kappa}$.

Requiring $K_r(s)$ to be a good approximation to $K(s)$, (say, as might occur if we conceived of the K -to- K_r reduction as simply an open-loop model reduction problem) is often not enough to preserve the desired closed-loop performance and effective controller reduction requires that plant dynamics be taken into account. This is generally achieved through frequency weighting [1], [10], [9]: Given a stabilizing controller $K(s)$, find a reduced-order controller

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$K_r(s)$ with the same number of unstable poles and so that the weighted error

$$\|W_o(s)(K(s) - K_r(s))W_i(s)\|_{\mathcal{H}_\infty}$$

is minimized. To ensure closed-loop stability, one chooses

$$W_i(s) = I, W_o(s) = [I + G(s)K(s)]^{-1}G(s), \text{ or} \quad (1)$$

$$W_o(s) = I, W_i(s) = G(s)[I + G(s)K(s)]^{-1}. \quad (2)$$

It is known [1], [11] that if either

$$\|[K(s) - K_r(s)]G(s)[I + K(s)G(s)]^{-1}\|_{\mathcal{H}_\infty} < 1, \text{ or} \quad (3)$$

$$\|[I + K(s)G(s)]^{-1}G(s)[K(s) - K_r(s)]\|_{\mathcal{H}_\infty} < 1, \quad (4)$$

then $K_r(s)$ is a stabilizing controller. On the other hand, to preserve closed-loop performance, one uses a two-sided weighting of the form

$$\begin{aligned} W_i(s) &= [I + G(s)K(s)]^{-1}, \text{ and} \\ W_o(s) &= [I + G(s)K(s)]^{-1}G(s) \end{aligned} \quad (5)$$

The above three weighted model reduction problems (1),(2), and (5) are usually approached with Enns' frequency-weighted balanced truncation method [3], which requires solving two large-scale Lyapunov equations of order n_κ or $n + n_\kappa$. In many cases, such as might be found in the control of systems modeled with PDEs, or in the control of large, flexible structures, n (and n_κ) easily reach into the tens of thousands and solving the associated Lyapunov equations becomes a fearsome computational task.

We propose here the use of a rational Krylov method [6] to reduce $K(s)$. We show that, besides being computationally more efficient relative to (frequency-weighted) balanced truncation techniques mentioned above, the reduced-order controller will provide a closed-loop system response that is guaranteed to match the full-order closed loop system response at selected interpolation points.

The rest of the paper is organized as follows: In Section II, we briefly discuss the rational Krylov method for model reduction, and introduce our proposed approach. Section III provides an illustrative numerical example, followed by conclusions in Section IV.

II. CONTROLLER REDUCTION VIA KRYLOV PROJECTION

Given a transfer function $H(s) = \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array} \right]$ and some selected interpolation points σ_i in the complex plane, the rational Krylov method generates a reduced order model

$$H_r(s) = \left[\begin{array}{c|c} \hat{A}_H & \hat{B}_H \\ \hline \hat{C}_H & \hat{D}_H \end{array} \right] \text{ that interpolates } H(s) \text{ as well}$$

as a certain number of its derivatives (called moments) at selected points σ_i , i.e.,

$$\left. \frac{(-1)^j}{j!} \frac{d^j H(s)}{ds^j} \right|_{s=\sigma_k} = \left. \frac{(-1)^j}{j!} \frac{d^j H_r(s)}{ds^j} \right|_{s=\sigma_k} \quad (6)$$

for $k = 1, \dots, K$ and for $j = 1, \dots, j_k$. Here K is the number of interpolation points σ_k and j_k is the number of moments to be matched at σ_k .

For a matrix $F \in \mathbb{C}^{n \times n}$, a vector $g \in \mathbb{C}^n$, and a point $\sigma \in \mathbb{C}$, we define the Krylov space of index j :

$$\begin{aligned} \mathcal{K}_j(F, g; \sigma) &:= \text{Im}([g \ Fg \ F^2g \ \dots \ F^{j-1}g]) \\ &\quad \text{if } \sigma = \infty \\ \mathcal{K}_j(F, g; \sigma) &:= \text{Im}([\ (\sigma I - F)^{-1}g \ \dots \ (\sigma I - F)^{-j}g \]) \\ &\quad \text{if } \sigma \neq \infty \end{aligned}$$

The following result, presented in [6], shows how to construct rational interpolants via Krylov projection:

Proposition 1: If

$$\text{Im}(V) = \text{Im}[\mathcal{K}_{j_1}(A, B; \sigma_1) \ \dots \ \mathcal{K}_{j_K}(A, B; \sigma_K)] \text{ and } \text{Im}(Z) =$$

$$\text{Im}[\mathcal{K}_{j_{K+1}}(A^T, C^T; \sigma_{K+1}) \ \dots \ \mathcal{K}_{j_{2K}}(A^T, C^T; \sigma_{2K})]$$

with $Z^T V = I$, then the reduced order model $H_r(s) = \left[\begin{array}{c|c} Z^T A_H V & Z^T B_H \\ \hline C_H V & D_H \end{array} \right]$ matches j_k moments of $H(s)$ at the interpolation point σ_k for each $k = 1, \dots, 2K$, i.e. $H_r(s)$ interpolates $H(s)$ and its first $j_k - 1$ derivatives at σ_k .

Proposition 1 states that for Krylov-based model reduction, all one has to do is to construct matrices V and Z as above. Efficient implementations of the rational Krylov method can be found in [6]. Unlike balanced truncation, Krylov-based model reduction methods require only matrix-vector multiplications and some sparse linear solves. They are able to maintain computational effectiveness even for large-scale problems; for more details see [4], [5], [6].

A. Controller reduction: The proposed approach

Given a plant $G(s)$ and a stabilizing controller $K(s)$, we propose to reduce the order of the controller using a rational Krylov method. The following proposition is the main result of this paper.

Proposition 2: For a given plant $G(s)$ and full-order controller $K(s)$, let $T(s)$ denote the full-order closed loop system: $T(s) = [1 + G(s)K(s)]^{-1}G(s)$. Given a set of interpolation points σ_k , $k = 1, \dots, 2K$, and the number of moments j_k to be matched at each σ_k , let $K_r(s)$ be the reduced r^{th} -order controller obtained from $K(s)$ by a rational Krylov method as outlined above. Denote by $T_r(s)$ the closed-loop system incorporating the reduced-order controller, K_r , in place of $K(s)$: $T_r(s) = [1 + G(s)K_r(s)]^{-1}G(s)$. Then $T_r(s)$ interpolates the full-order closed loop system $T(s)$ and its first $j_k - 1$ derivatives at σ_k for $k = 1, \dots, 2K$, i.e.,

$$\left. \frac{(-1)^j}{j!} \frac{d^j T(s)}{ds^j} \right|_{s=\sigma_k} = \left. \frac{(-1)^j}{j!} \frac{d^j T_r(s)}{ds^j} \right|_{s=\sigma_k} \quad (7)$$

for $k = 1, \dots, 2K$ and for $j = 1, \dots, j_k$.

Proposition 2 states that a reduced-order controller obtained through a rational Krylov reduction as described is *guaranteed* to yield closed-loop behavior which approximates the full-order closed-loop system, at least in the neighborhood of selected frequencies. Moreover, the procedure involved in obtaining this reduced-order controller is suitable for large-scale settings where $n_{rc} \gg 1000$.

B. Selection of interpolation points σ_i

We suggest two ways for choosing interpolation points. The first choice is based on a recent result [8], whereas the second one is based on equations (3), (4) and (5) above.

- 1) In [8], Gugercin and Antoulas showed that the \mathcal{H}_2 error in model reduction is due to mismatch of original and reduced order transfer functions at the mirror images of the full-order and reduced-order poles. Based on this observation, they proposed choosing mirror images of the full-order poles as interpolation points. For the controller reduction problem, we make a similar choice, taking σ_k from the union of mirror images of the poles of $T(s)$ and $K(s)$. This choice is expected to yield a small \mathcal{H}_2 error for both error systems $K(s) - K_r(s)$ and $T(s) - T_r(s)$. Indeed as in the (open-loop) model reduction case [8], this choice seems to work efficiently for controller reduction problem as well.
- 2) It follows from equations (3), (4) and (5) that $K_r(s)$ must approximate $K(s)$ well wherever $|W_i(j\omega)|$ and $|W_o(j\omega)|$ are large. Hence, based on this observation, one may choose σ_i in frequency intervals on the imaginary axis where $|W_i(j\omega)|$ and $|W_o(j\omega)|$ are large.

Remark 1: The controller $K(s)$ in general does not need to be stable. If it does fail to be stable then, as suggested in [10], one may decompose $K(s)$ as

$$K(s) = K_-(s) + K_+(s),$$

where $K_-(s)$ and $K_+(s)$ denote the stable and anti-stable parts of $K(s)$, respectively. Then model reduction by rational Krylov is applied to the stable part $K_-(s)$ only, leaving the anti-stable part $K_+(s)$ unreduced. The moments of $K(s)$ are equal to the summation of moments of $K_-(s)$ and $K_+(s)$, and since we match moments of $K_-(s)$ and retain $K_+(s)$ in its entirety, the key interpolation property in (7) still holds. For stability of the closed-loop system, $K(s)$ and $K_r(s)$ must have the same number of unstable poles. The two selection criteria proposed above seem to work very efficiently in practice, yet may cause a loss of stability in $K_r(s)$. One may use implicit restarting [7] to regain stability, though this comes at the cost of perturbing the interpolation conditions of (7).

III. EXAMPLE

The full order model considered is a model of the Los Angeles University Hospital building with 8 floors each of

which has 3 degrees of freedom, namely displacements in x and y directions, and rotation. In the state-space, the model has order $n = 48$, and is single input and single output. The system is lightly damped, i.e., has pole close to imaginary axis, and impulse response shows long-lasting oscillations which are not desired. The goal here is to design a controller to move those poles away from imaginary so that oscillations are substantially smaller in magnitude and shorter in duration. A state-feedback observer-based full-order controller was designed to meet the desired closed loop performance so that the pole closest to the imaginary axis had a real part of -0.27 . The impulse responses of the open loop ($G(s)$) and closed loop ($T(s)$) systems are shown in Figures 1-(a) and 1-(b), respectively. Observe that for an impulse input (representing an earthquake), the open loop system keeps oscillating up to 15 seconds, while the closed loop system stabilizes within 5 seconds, with smaller amplitudes throughout. This initial controller has the

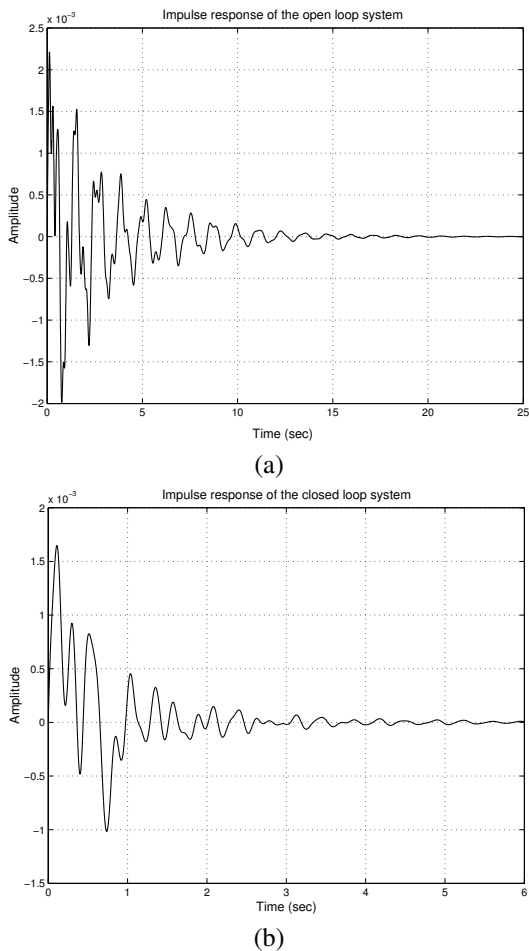


Fig. 1. The impulse responses of (a) open loop system $G(s)$, (b) closed loop system $T(s)$

same complexity as the plant; its order is $n_c = 48$. Using the rational Krylov approach proposed above, a reduced-order controller with order $r = 18$ was calculated. Figure 2 depicts the impulse responses of both $T(s)$ and $T_r(s)$.

The response of $T_r(s)$ matches that of $T(s)$ almost exactly. We also include the sigma plots of $T(s)$ and $T_r(s)$ in Figure 3, which shows that $T_r(s)$ is a very good match to $T(s)$. The \mathcal{H}_∞ and \mathcal{H}_2 errors between $T(s)$ and $T_r(s)$ were computed as: $\|T(s) - T_r(s)\|_{\mathcal{H}_\infty} = 2.84 \times 10^{-5}$ and $\|T(s) - T_r(s)\|_{\mathcal{H}_2} = 4.74 \times 10^{-5}$. These results underscore the effectiveness of the proposed approach.

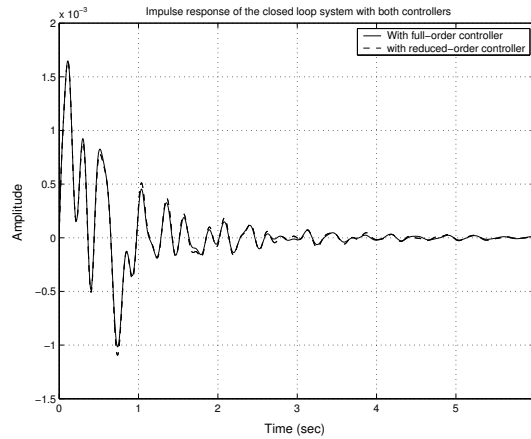


Fig. 2. The impulse responses of the full-order and reduced-order closed loop systems

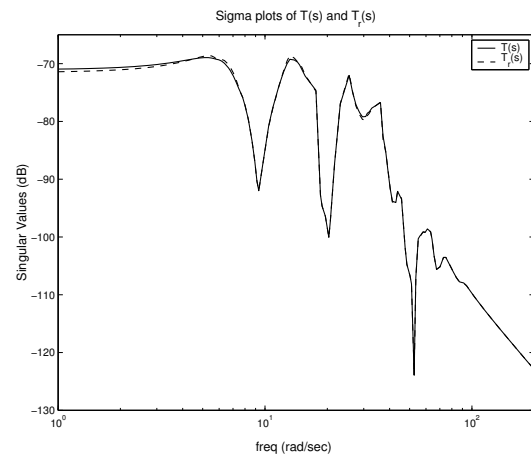


Fig. 3. The sigma plots of the full-order ($T(s)$) and reduced-order closed loop systems ($T_r(s)$)

IV. CONCLUSION

In this note, we have introduced a Krylov-based controller reduction method. Not only the reduced-order controller matches the full-order controller, but also the resulting reduced-order closed loop system matches the full-order closed loop at the selected interpolation points. Moreover, unlike the frequency weighted balanced reduction approach for controller reduction, the proposed method does not require solving any Lyapunov equations and uses computational tools that scale well to large-order problems. A numerical example illustrates the effectiveness of our approach.

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